On the Calculation of Nonlinear Spinor Field Functionals. II

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In preceding papers a functional quantum theory has been developped. In this theory the physical observables are derived only by functional operations from the corresponding state functionals. To obtain physical informations therefore the state functionals have to be known and calculational procedures have to be developped. As high energy phenomena are of interest the state functionals of the nonlinear spinor field are considered. In a preceding paper I a calculation method for bound state functionals has been proposed. In this paper this method is extended to the calculation of scattering state functionals, resulting in channel equations for the various scattering processes in the dynamics of the nonlinear spinor field.

Introduction

Nonlinear spinor field theory with noncanonical relativistic Heisenberg quantization is an approach to a unified microscopic description of matter [1]. As the spinorfield operators cannot be identified with operators of free matter fields a special theory of observables is required to obtain meaningfull physical informations from them. Concerning the global observables of the quantized spinorfield, this theory is provided by the functional quantum theory of the nonlinear spinor field, which has been proposed by Stumpf [2] and has been developped in a series of papers by Stumpf and coworkers [3, 4]. To obtain numerical results comparable with experiment by the functional quantum theory the eigenstates of the nonlinear spinor field in the corresponding functional space have to be calculated, i. e. the spinor field functional states especially bound functional states describing single particles and scattering functional states, describing the scattering of several particles. In a preceding paper [5] a functional formalism for the construction of functional bound states has been derived. In this paper we discuss the construction of functional scattering states. To avoid lengthy repetitions we refer frequently to the preceding papers, denoting paper [5], by I, and paper [3] by II. A first attempt to derive scattering equations for the nonlinear spinor field has been made by Dürr and Wagner [6] by a graphical discussion of the lowest order Tamm-Dancoff equations. But this approach was not sufficient to solve the problem properly as can be seen by a comparison with the discussion given here. Additionally the method presented here is not restricted to spinorfield functionals but can be applied to any type of field functionals, where the corresponding field dynamics contains relativistic clusters, i. e. bound states which act like elementary particles.

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1. Fundamentals

Summarizing the preceding investigations, the functional quantum theory of the nonlinear spinorfield is characterized

a) by one or several linear functional state spaces with a well defined physical functional scalar product,
b) by a maximal set of compatible dynamical functional equations, leading to physical functional eigenstates,
c) by a physical and probabilistic interpretation of these states provided by a map of the functional space into ordinary Hilbert space.

For brevity we use the general summation convention

$$g(z)f(z) := \sum_{\alpha} \int g_{\alpha}(z) f^{\alpha}(z)\, \mathrm{d}^4 z \qquad (1.1)$$

in the following. Then the functional eigenstates are given by the time ordered functional states II (2.9), which can be written generally

$$|\mathfrak{T}(j)\rangle := \sum_{n=1}^{\infty} \tau_n(x_1 \ldots x_n)\, |D_n(x_1 \ldots x_n)\rangle \,. \qquad (1.2)$$

where the set $\{|D_n\rangle\}$ defines the base functionals of the functional space. The quantum numbers characterizing the functional eigenstates due to the relativistic invariance of the theory under the Poincaré

group are given by the conditions, i. e. dynamical equations

$$\begin{aligned} \mathfrak{P}_h \,|\, \mathfrak{T}(j)\rangle &= \mathfrak{J}_h \,|\, \mathfrak{T}(j)\rangle ; \\ \mathfrak{G}_\mu \mathfrak{G}^\mu \,|\, \mathfrak{T}(j)\rangle &= s(s+1)\,|\, \mathfrak{T}(j)\rangle ; \\ \mathfrak{P}^2 \,|\, \mathfrak{T}(j)\rangle &= m^2 \,|\, \mathfrak{T}(j)\rangle , \\ \mathfrak{S}_3 \,|\, \mathfrak{T}(j)\rangle &= s_3 \,|\, \mathfrak{T}(j)\rangle . \end{aligned} \tag{1.3}$$

Additionally a functional dynamical equation resulting from the nonlinear spinor field equation has to be satisfied. In its original form II (2.21) this equation does not commute with the permutation group operators. As the true physical eigenstates (1.2) have to be representation states of this group too, the proper dynamical equation has to be form-invariant under this group. This can be achieved by a suitable symmetrization [7]. Assuming for brevity for the noncanonical quantized spinor field $F(0) = \varrho_0 \equiv 0$ and applying $j_\alpha(x)\,(-i\,\partial_\mu)$ to II (2.21) by means of (1.3) the equation

$$[\gamma^\mu \mathfrak{P}_\mu + \mathfrak{D}(j, \partial)]\,|\, \mathfrak{T}(j)\rangle = 0 \tag{1.4}$$

with

$$\begin{aligned} \mathfrak{D}(j, \partial) :&= -i\,\gamma^\mu j(x)\,\partial_\mu\, G(x-x') \\ &\cdot V(x', y, z, u)\,\partial(y)\,\partial(z)\,\partial(u) \end{aligned} \tag{1.5}$$

results, where $V(x, y, z, u)$ is the general vertex of the spinor field interaction term given in [6]. To define the noncanonical quantization properly, the normal transform of (1.2)

$$|\, \Phi(j)\rangle :\, = \exp[j(x) F(x, y)\, j(y)]\,|\, \mathfrak{T}(j)\rangle \tag{1.6}$$

has to be used in the following. It can be written generally

$$|\, \Phi(j)\rangle :\, = \sum_{n=1}^{\infty} \varphi_n(x_1 \ldots x_n)\,|\, D_n(x_1 \ldots x_n)\rangle \tag{1.7}$$

and its use is no restriction with respect to calculations in conventional quantized theories. Under the normal transformation the equations (1.3) remain invariant giving so

$$\begin{aligned} \mathfrak{P}_h \,|\, \Phi(j)\rangle &= \mathfrak{J}_h \,|\, \Phi(j)\rangle , \\ \mathfrak{G}_\mu \mathfrak{G}^\mu \,|\, \Phi(j)\rangle &= s(s+1)\,|\, \Phi(j)\rangle , \\ \mathfrak{P}^2 \,|\, \Phi(j)\rangle &= m^2 \,|\, \Phi(j)\rangle , \\ \mathfrak{S}_3 \,|\, \Phi(j)\rangle &= s_3 \,|\, \Phi(j)\rangle \end{aligned} \tag{1.8}$$

while the dynamical Eq. (1.4) goes over into

$$[\gamma^\mu \mathfrak{P}_\mu + \mathfrak{D}(j, \mathrm{d})]\,|\, \Phi(j)\rangle = 0 \tag{1.9}$$

where d is defined by II. (2.24).

By means of (1.8) this equation can be changed into an eigenvalue equation for m^2 as was done in I. For the present purpose it is more convenient to use only the momentum average as was done in [8] for eigenvalue calculations. This leads to the eigenstate equation

$$[\gamma^\mu \mathfrak{J}_\mu - \mathfrak{D}(j, d)]\,|\, \Phi(j)\rangle = 0 . \tag{1.10}$$

As additional discrete symmetry groups influence the general discussion performed in this paper only by a definite statement being discussed later, we omit for brevity the conditions resulting by these groups. Denoting generally the set of good quantum numbers by a, we write for the corresponding eigenstate $|\, \Phi(j)\rangle \equiv |\, \Phi(j, a)\rangle$. By the map of the physical functional eigenstates into ordinary Hilbert-space of the theory, it is obvious, that two different types of eigenstates have to occur, namely

α) bound states, β) scattering states.

These two types of states are distinguished by different boundary conditions to be satisfied. In order to derive proper eigenstates an important task is the incorporation of the boundary conditions into the solution procedure. It is therefore convenient to distinguish between steps in the solution procedure which are generally valid and those which depend on the boundary conditions. We discuss first that part of the solution procedure which is generally valid, i. e. does not depend on bound states or scattering states in consideration. We define the projection operator

$$\begin{aligned} P_k :\, &= |\, D_k(z_1 \ldots z_k)\rangle\, \delta(z_1 - y_1) \ldots \delta(z_k - y_k) \\ &\cdot \langle D_k(y_1 \ldots y_k)|\, . \end{aligned} \tag{1.11}$$

For a given eigenstate $|\, \Phi(j, a)\rangle$ we consider the projections $P_k\,|\, \Phi(j, a)\rangle$, $1 \leqq k < \infty$. Then we look for the smallest k, for which $P_k\,|\, \Phi(j, a)\rangle \neq 0$. Denoting this by $k = \varrho$, we have $P_k\,|\, \Phi(j, a)\rangle = 0$, $1 \leqq k \leqq \varrho - 1$ and

$$|\, \Phi_\varrho(j, a)\rangle :\, = P_\varrho\,|\, \Phi(j, a)\rangle , \tag{1.12}$$

and the eigenstate can be written

$$|\, \Phi(j, a)\rangle = |\, \Phi_\varrho(j, a)\rangle + |\, \Phi_r(j, a)\rangle \tag{1.13}$$

with

$$|\, \Phi_r(j, a)\rangle :\, = \Pi_\varrho\,|\, \Phi(j, a)\rangle \tag{1.14}$$

and

$$\Pi_\varrho :\, = \sum_{k=\varrho+1}^{\infty} P_k . \tag{1.15}$$

Without proof we state here, that the quantum numbers resulting from all symmetry groups of the theory fix the smallest $k = \varrho$ for a given set of these numbers uniquely. So no indefiniteness concerning

the introduction of ϱ is possible. Then it was demonstrated in I, that for $|\Phi_\varrho(j,a)\rangle$ the equation

$$[(\mathfrak{J}-P_\varrho\,\mathfrak{D}\,P_\varrho)-P_\varrho(\mathfrak{J}-\mathfrak{D})\Pi_\varrho\,(\mathfrak{J}-\Pi_\varrho\,\mathfrak{D}\,\Pi_\varrho)^{-1}\cdot\Pi_\varrho(\mathfrak{J}-\mathfrak{D})P_\varrho]\,|\Phi_\varrho(j,a)\rangle=0 \quad (1.16)$$

with $\mathfrak{J}:=\gamma^\mu\mathfrak{J}_\mu$ can be derived from (1.10). This equation can be solved independently from $|\Phi_r(j,a)\rangle$, which is considerable progress concerning the solution procedure, as in configuration space the solution $|\Phi_\varrho(j,a)\rangle$ is a function of only ϱ independent Minkowsky coordinates. Solving (1.16) for $|\Phi_\varrho(j,a)\rangle$ the complete functional eigenstate can be derived by observing

$$|\Phi_r(j,a)\rangle=-(\mathfrak{J}-\Pi_\varrho\,\mathfrak{D}\,\Pi_\varrho)^{-1}\cdot\Pi_\varrho(\mathfrak{J}-\mathfrak{D})\,P_\varrho\,|\Phi_\varrho(j,a)\rangle\,. \quad (1.17)$$

In configuration space the Eq. (1.16) is an integral equation. For the calculation of bound states the boundary conditions are defined by those of the kernel of the equation itself. So solving the homogeneous Eq. (1.16) for bound states the proper boundary conditions are achieved immediately. For scattering states the boundary conditions are not given by those of the kernel, but by the ingoing and outgoing configurations. Therefore in a second step we have to prepare a solution procedure concerning these conditions. Nevertheless it is generally valid, that (1.16) can be used for bound states as well as for scattering states. So we have for the calculation of scattering states not to treat the full Eq. (1.10) but only Eq. (1.16), i.e. this equation is a necessary condition to be satisfied by scattering states, too. In the following we shall treat Eq. (1.16) only in order to obtain a sufficient condition for scattering functionals.

2. Channel Equations

To incorporate boundary conditions into the dynamical equation we consider first the scattering functionals. The quantum numbers a of scattering functionals are given by the quantum numbers of the corresponding ingoing or outgoing asymptotic configurations, which shall be proven in Section 3. By the map into ordinary Hilbert space one concludes that for a scattering functional $|\Phi^{(\pm)}(j,a)\rangle$ the decomposition

$$|\Phi^{(\pm)}(j,a)\rangle=|\Phi(j,a)\rangle^c+|\chi^{(\pm)}(j,a)\rangle \quad (2.1)$$

has to be valid, where $|\Phi(j,a)\rangle^c$ is a free cluster state

$$|\Phi(j,a)\rangle^c\equiv|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle:=\mathfrak{A}^+(\mathfrak{K}_1)\ldots\mathfrak{A}^+(\mathfrak{K}_n)\,|\varphi_0\rangle \quad (2.2)$$

defining the ingoing resp. outgoing configuration of elementary particles resp. bound states, denoted by the quantum numbers $\mathfrak{K}_1\ldots\mathfrak{K}_n$. The states

$$|\Phi(j,\mathfrak{K}_i)\rangle:=\mathfrak{A}^+(\mathfrak{K}_i)\,|\varphi_0\rangle\,,\ 1\leqq i\leqq n\,, \quad (2.3)$$

are eigenstates of the theory, i.e. they satisfy the equations

$$[\mathfrak{J}(i)-\mathfrak{D}(j,d)]\,|\Phi(j,\mathfrak{K}_i)\rangle=0\,,\ 1\leqq i\leqq n\,, \quad (2.4)$$

together with

$$\begin{aligned}\mathfrak{P}_h\,|\Phi(j,\mathfrak{K}_i)\rangle&=\mathfrak{J}_h(i)\,|\Phi(j,\mathfrak{K}_i)\rangle\,,\\ \mathfrak{P}^2\,|\Phi(j,\mathfrak{K}_i)\rangle&=m_i^2\,|\Phi(j,\mathfrak{K}_i)\rangle\,,\\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\Phi(j,\mathfrak{K}_i)\rangle&=s_i(s_i+1)\,|\Phi(j,\mathfrak{K}_i)\rangle\,,\\ \mathfrak{S}_3\,|\Phi(j,\mathfrak{K}_i)\rangle&=s_{3i}\,|\Phi(j,\mathfrak{K}_i)\rangle\,.\end{aligned} \quad (2.5)$$

Concerning the free many particle states (2.2) it can be shown that they do not satisfy the dynamical Eq. (1.10), but only the symmetry conditions (1.8), i.e. the states (2.2) are no dynamical eigenstates of the theory. By direct calculation follows

$$\begin{aligned}\mathfrak{P}_h\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=\mathfrak{J}_h\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{P}^2\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=\mathfrak{J}_h\,\mathfrak{J}^h\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=s(s+1)\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{S}_3\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=s_3\,|\Phi(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\end{aligned} \quad (2.6)$$

with

$$\mathfrak{J}_h:=\sum_{i=1}^{n}\mathfrak{J}_h(i)\,;\quad s=\sum_{i=1}^{n}s_i\,. \quad (2.7)$$

For the scattering states (2.1) in contrary the equations

$$[\mathfrak{J}-\mathfrak{D}(j,d)]\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle=0\,, \quad (2.8\,a)$$

$$\begin{aligned}\mathfrak{P}^h\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=\mathfrak{J}_h\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{P}^2\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=\mathfrak{J}_h\,\mathfrak{J}^h\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=s(s+1)\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{S}_3\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle&=s_3\,|\Phi^{(\pm)}(j,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\end{aligned} \quad (2.8\,b)$$

are satisfied with the same quantum numbers (2.7), i. e. these states are true eigenstates of the theory. This will be shown in Section 3.

Now we return to the general solution procedure of Section 1. As was already emphasized this procedure is valid for bound states as well as for scattering states. To incorporate the appropriate boundary conditions into Eq. (1.16), we apply an auxiliary construction in configuration space. We define the configuration space representation

$$K_\varrho(x_1 \dots x_\varrho, y_1 \dots y_\varrho): \tag{2.9}$$
$$= \langle D_\varrho(x_1 \dots x_\varrho) | \, [P_\varrho \mathfrak{D} P_\varrho + P_\varrho(\mathfrak{J} - \mathfrak{D}) \Pi_\varrho (\mathfrak{J} - \Pi_\varrho \mathfrak{D} \Pi_\varrho)^{-1} \Pi_\varrho (\mathfrak{J} - \mathfrak{D}) P_\varrho] \, | D_\varrho(y_1 \dots y_\varrho) \rangle$$

and
$$\varphi_\varrho(x_1 \dots x_\varrho \,|\, a) := \langle D_\varrho(x_1 \dots x_\varrho) | \, \Phi_\varrho(j, a) \rangle \,. \tag{2.10}$$

Then Eq. (1.11) can be written

$$[\mathfrak{J}\, \delta(x_1 - y_1) \dots \delta(x_\varrho - y_\varrho) + K_\varrho(x_1 \dots x_\varrho \,|\, y_1 \dots y_\varrho)] \, \varphi_\varrho(y_1 \dots y_\varrho \,|\, a) = 0 \tag{2.11}$$

or symbolically
$$[\mathfrak{J} \mathbf{1}_\varrho + \mathbf{K}_\varrho] \, \varphi_\varrho(a) = 0 \,. \tag{2.12}$$

As the states (2.3) are true eigenstates, their projections $P_{\varrho_i} | \, \Phi(j, \mathfrak{K}_i) \rangle \equiv | \Phi_{\varrho_i}(j, \mathfrak{K}_i) \rangle$

with
$$\varphi_{\varrho_i}(x_1 \dots x_{\varrho_i} \,|\, \mathfrak{K}_i) := \langle D_{\varrho_i}(x_1 \dots x_{\varrho_i}) | \, \Phi_{\varrho_i}(j, \mathfrak{K}_i) \rangle \,, \quad 1 \leqq i \leqq n \,, \tag{2.13}$$

satisfy in configuration space the equations

$$[\mathfrak{J}(i) \mathbf{1}_{\varrho_i} + \mathbf{K}_{\varrho_i}] \, \varphi_{\varrho_i}(\mathfrak{K}_i) = 0 \,, \quad 1 \leqq i \leqq n \,. \tag{2.14}$$

Considering now the projection

$$\varphi_\varrho^{(\pm)}(x_1 \dots x_\varrho \,|\, \mathfrak{K}_1 \dots \mathfrak{K}_n) := \langle D_\varrho(x_1 \dots x_\varrho) | \, \Phi_\varrho^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle \tag{2.15}$$

with $\varrho = \sum_{i=1}^{n} \varrho_i$ we obtain immediately from (2.1)

$$\varphi_\varrho^{(\pm)}(x_1 \dots x_\varrho \,|\, \mathfrak{K}_1 \dots \mathfrak{K}_n) = \mathcal{A}[\varphi_{\varrho_1}(x_1 \dots x_{\varrho_1} \,|\, \mathfrak{K}_1) \dots \varphi_{\varrho_n}(x_a \dots x_\varrho \,|\, \mathfrak{K}_n)] + \chi_\varrho^{(\pm)}(x_1 \dots x_\varrho \,|\, \mathfrak{K}_1 \dots \mathfrak{K}_n) \tag{2.16}$$

where $\mathcal{A}$ means the complete antisymmetrization of the product of functions φ_{ϱ_i}, $1 \leqq i \leqq n$, which are themselves already antisymmetrized. Without any inconsistency we may assume

$$P_k | \, \Phi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = 0 \,, \quad 1 \leqq k \leqq \varrho - 1 \quad \text{as} \quad P_k | \, \Phi(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = 0 \,, \quad 1 \leqq k \leqq \varrho - 1 \,.$$

Observing (2.8) the general procedure of Section 1 applied to (2.1) therefore gives the equation

$$[\mathfrak{J} \mathbf{1}_\varrho + \mathbf{K}_\varrho] \, \varphi_\varrho^{(\pm)}(\mathfrak{K}_1 \dots \mathfrak{K}_n) = 0 \tag{2.17}$$

which has to be satisfied by (2.16). By (2.16) also the boundary conditions can be seen, which have to be satisfied by a solution of (2.17). To incorporate them into (2.17) we use a method described by Goldberger and Watson for nonrelativistic many channel problems [9]. This method uses an intermediate step, where the full antisymmetry of the solutions is given up, and where it is restored only in a final step. To apply this method we observe, that the fully antisymmetrized product of ingoing resp. outgoing free cluster wavefunction (2.12) can be written

$$\mathcal{A}[\varphi_{\varrho_1} \dots \varphi_{\varrho_n}] = \sum_{\varkappa=1}^{L} Q_\varkappa \, \varphi_{\varrho_1} \dots \varphi_{\varrho_n} \tag{2.18}$$

where $Q_\varkappa$ denotes the operator of the permutation $\varkappa$. As the single wavefunctions φ_{ϱ_i}, $1 \leqq i \leqq n$ of the clusters themselves are already fully antisymmetrized in the coordinates, the $Q_\varkappa$ are those permutations operations which are required additionally to complete the antisymmetrization of the product (2.18). By definition we consider virtual scattering states, which refer not to the completely antisymmetrized initial configuration (2.16), but only to one definite permutation $Q_\varkappa$. This gives the solutions

$$\varphi_\varrho^{(\pm)}(x_1 \dots x_\varrho \,|\, \varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n) = Q_\varkappa \, \varphi_{\varrho_1}(x_1 \dots x_{\varrho_1} \,|\, \mathfrak{K}_1) \dots \varphi_{\varrho_n}(x_a \dots x_\varrho \,|\, \mathfrak{K}_n) + \chi_\varrho^{(\pm)}(x_1 \dots x_\varrho \,|\, \varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n) \,. \tag{2.19}$$

Also by definition we require

$$[\mathfrak{J} \mathbf{1}_\varrho + \mathbf{K}_\varrho] \, \varphi_\varrho^{(\pm)}(\varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n) = 0 \,, \; 1 \leqq \varkappa \leqq L \,. \tag{2.20}$$

Although the kernel of this equation is invariant under permutation operations, the solution $\varphi_\varrho^{(\pm)}(\varkappa)$ is no representation state of this group. But this is no contradiction as a solution is determined not only by the kernel itself but also by boundary conditions. Then due to the linearity of Eq. (2.17) it is clear, that the true physical solution (2.16) of Eq. (2.17) can be decomposed into the series of virtual solutions (2.19) giving so

$$\varphi_\varrho^{(\pm)}(\mathfrak{K}_1 \dots \mathfrak{K}_n) = \sum_\varkappa \varphi_\varrho^{(\pm)}(\varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n) . \tag{2.21}$$

Therefore the physical solution can be obtained, if one is able to calculate the virtual solutions (2.19). To achieve this we define

$$\mathbf{K}_\varrho(\varrho_1 \dots \varrho_n) : = \mathbf{K}_{\varrho_1}(x_1 \dots x_{\varrho_1}, y_1 \dots y_{\varrho_1}) + \dots + \mathbf{K}_{\varrho_n}(x_\alpha \dots x_\varrho, y_\alpha \dots y_\varrho) . \tag{2.22}$$

Due to $\mathfrak{J} = \sum_{i=1}^{n} \mathfrak{J}(i)$ it follows from (2,14), that the equation

$$[\mathfrak{J}\,\mathbf{1}_\varrho + \mathbf{K}_\varrho(\varrho_1 \dots \varrho_n)]\, \varphi_{\varrho_1}(y_1 \dots y_{\varrho_1} \mid \mathfrak{K}_1) \dots \varphi_{\varrho_n}(y_\alpha \dots y_\varrho \mid \mathfrak{K}_n) = 0 \tag{2.23}$$

is satisfied. As for any permutation $Q_\varkappa$ the inverse element $Q_\varkappa^{-1}$ with $Q_\varkappa^{-1} Q_\varkappa = 1$ exists, due to the group structure of permutations, from (2.23) follows immediately that

$$[\mathfrak{J}\,\mathbf{1}_\varrho + Q_\varkappa \mathbf{K}_\varrho(\varrho_1 \dots \varrho_n)\, Q_\varkappa^{-1}]\, Q_\varkappa \varphi_{\varrho_1} \dots \varphi_{\varrho_n} = 0, \quad 1 \leqq \varkappa \leqq L \tag{2.24}$$

is satisfied also. Defining now the channel interaction operator

$$\mathbf{V}_\varrho(\varkappa) : = \mathbf{K}_\varrho - Q_\varkappa \mathbf{K}_\varrho(\varrho_1 \dots \varrho_n) Q_\varkappa^{-1}, \; 1 \leqq \varkappa \leqq L \tag{2.25}$$

Eq. (2.20) can be written

$$[\mathfrak{J}\,\mathbf{1}_\varrho + Q_\varkappa \mathbf{K}_\varrho(\varrho_1 \dots \varrho_n)\, Q_\varkappa^{-1} + \mathbf{V}_\varrho(\varkappa)]\; \varphi_\varrho^{(\pm)}(\varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n) = 0 , \, 1 \leqq \varkappa \leqq L . \tag{2.26}$$

But from this equation follows by inversion the channel equation

$$\varphi_\varrho^{(\pm)}(\varkappa, \mathfrak{K}_1 \dots \mathrm{K}_n) = Q_\varkappa \varphi_{\varrho_1}(\mathfrak{K}_1) \dots \varphi_{\varrho_n}(\mathfrak{K}_n) + \lim_{\gamma \to 0} [\mathfrak{J}\,\mathbf{1}_\varrho + Q_\varkappa \mathbf{K}_\varrho(\varrho_1 \dots \varrho_n)\, Q_\varkappa^{-1} \pm i\,\gamma]^{-1} \cdot \mathbf{V}_\varrho(\varkappa) \varphi_\varrho^{(\pm)}(\varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n) , \; 1 \leqq \varkappa \leqq L . \tag{2.27}$$

Assuming (2.27) to be solved by standard methods, the virtual scattering states $\varphi_\varrho^{(\pm)}(\varkappa, \mathfrak{K}_1 \dots \mathfrak{K}_n)$ can be obtained, satisfying the required boundary conditions. But then by (2.21) immediately the true physical scattering states $\varphi_\varrho^{(\pm)}(\mathfrak{K}_1 \dots \mathfrak{K}_n)$ are obtained.

3. Scattering Functionals

In the preceding section the quantities

$$\varphi_\varrho^{(\pm)}(\mathfrak{K}_1 \dots \mathfrak{K}_n) = \langle D_\varrho(x_1 \dots x_\varrho) \mid \Phi_\varrho^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle \tag{3.1}$$

have been derived i. e. can be assumed to be calculable. Therefore one obtains for $P_\varrho \mid \Phi^{(\pm)}(j\,\mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle$

$$P_\varrho \mid \Phi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = \varphi_\varrho^{(\pm)}(x_1 \dots x_\varrho \mid \mathfrak{K}_1 \dots \mathfrak{K}_n) \mid D_\varrho(x_1 \dots x_\varrho) \rangle . \tag{3.2}$$

Now by (1.17) follows

$$\mid \Phi_r^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = - (\mathfrak{J} - \Pi_\varrho \mathfrak{D}\, \Pi_\varrho)^{-1}\, \Pi_\varrho (\mathfrak{J} - \mathfrak{D}) P_\varrho \mid \Phi_\varrho^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle \tag{3.3}$$

and by (1.13)

$$\mid \Phi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = \mid \Phi_\varrho^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle + \mid \Phi_r^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle . \tag{3.4}$$

This has to be compared with the general decomposition (2.1) exhibiting the boundary conditions

$$\mid \Phi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = \mid \Phi(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle + \mid \chi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle . \tag{3.5}$$

Applying P_ϱ to (3.5) one obtains

$$P_\varrho \mid \Phi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = P_\varrho \mid \Phi(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle + P_\varrho \mid \chi^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle = \mid \Phi_\varrho^{(\pm)}(j, \mathfrak{K}_1 \dots \mathfrak{K}_n) \rangle \tag{3.6}$$

which indeed is satisfied by observing (2.16) which has been secured by construction. So the problem remains to show whether

$$\Pi_\varrho \,|\, \Phi^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle = |\, \Phi_r{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle = \Pi_\varrho \,|\, \Phi(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle + \Pi_\varrho \,|\, \chi^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \quad (3.7)$$

is valid or not, i. e. that $\Pi_\varrho \,|\, \Phi^{(\pm)}\rangle$ satisfies also the proper boundary conditions. To perform this, we assume that the division (3.7) leads to an expression $\Pi_\varrho |\chi^{(\pm)}\rangle$ which does not satisfy the proper boundary conditions. Then we introduce the test functional

$$|\, \tilde{\Phi}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle := |\, \tilde{\Phi}_\varrho{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle + |\, \tilde{\Phi}_r{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \quad (3.8)$$

where

$$|\, \tilde{\Phi}_\varrho{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \equiv |\, \Phi_\varrho{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \quad (3.9)$$

and

$$|\, \tilde{\Phi}_r{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle = \Pi_\varrho \,|\, \Phi(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle + \Pi_\varrho \,|\, \tilde{\chi}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \quad (3.10)$$

is valid, and which is assumed, to be a solution of the dynamical Eq. (1.10) with the proper boundary conditions. As Eq. (1.10) is a linear equation we may consider also the solutions

$$|\, Z^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle := |\, \Phi^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle - |\, \tilde{\Phi}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \quad (3.11)$$

for which we use the decomposition

$$|\, Z^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle = |\, Z_\varrho{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle + |\, Z_r{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle . \quad (3.12)$$

Then from (3.9) follows

$$|\, Z_\varrho{}^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \equiv 0 . \quad (3.13)$$

So $|\, Z^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle$ has to be a scattering solution of the Eq. (1.10) in the subspace of the projection operator Π_ϱ with the quantum numbers $\mathfrak{K}_1 \ldots \mathfrak{K}_n$. But this is a contradiction, as the quantum numbers $\mathfrak{K}_1 \ldots \mathfrak{K}_n$ require exactly the representation space of $(P_\varrho + \Pi_\varrho)$, so $|\, Z^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \equiv 0$ follows. Therefore by the construction used above the true scattering functional can be constructed uniquely.

Finally we show, that the true scattering functionals satisfy the symmetry conditions (2.8 b). We observe first, that the infinitesimal generators $\mathfrak{P}_h$, $\mathfrak{P}^2$, $\mathfrak{G}_\mu \mathfrak{G}^\mu$, $\mathfrak{S}_3$ commute with $\mathfrak{D}(j, d)$. This can be verified by direct calculation. Additionally it can be shown, that $\mathfrak{P}_h, \mathfrak{P}^2, \mathfrak{G}_\mu \mathfrak{G}^\mu, \mathfrak{S}_3$ commute also with $P_k \mathfrak{D}(j, d) P_l$, $1 \leqq k, l < \infty$. Considering now the Eq. (1.5) by standard procedures it can be brought into the form

$$|\, \Phi^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle = \lim_{\gamma \to 0} \pm i\gamma (\mathfrak{J} - \mathfrak{D} \pm i\gamma)^{-1} \,|\, \Phi(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle \quad (3.14)$$

if $|\, \Phi(j, a)\rangle$ is a scattering eigenstate. But from this representation follows immediately (2.8) if one observes (2.6) and the commutation relations of $\mathfrak{D}$ just mentioned. The same technique can be applied by considering the decomposition (3.4). Then the same statements can be proven for $|\, \Phi_\varrho{}^{(\pm)}(j, a)\rangle$ and $|\, \Phi_r{}^{(\pm)}(j, a)\rangle$ separately.

4. Alternative Channel Equations

For completeness we discuss a second method of derivation of appropriate channel equations, which does not refer to an analogy in nonrelativistic many channel problems. To perform this we symmetrize the original Eq. II (2.20) by functional integration as has been done in [8]. The result may be written

$$[1 + \hat{\mathfrak{D}}(j, d)]\,|\, \Phi(j)\rangle = 0 \quad (4.1)$$

where details can be found in [8]. Therefore for scattering calculations Eqs. (2.4) and (2.8 a) can be replaced equivalently by

$$[1 + \hat{\mathfrak{D}}(j, d)]\,|\, \Phi(j, \mathfrak{K}_i)\rangle = 0 , \; 1 \leqq i \leqq n , \quad (4.2)$$

and

$$[1 + \hat{\mathfrak{D}}(j, d)]\,|\, \Phi^{(\pm)}(j, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\rangle = 0 \quad (4.3)$$

where the operator $\hat{\mathfrak{D}}$ depends of course on the eigenvalues of the eigenstate in consideration. Using

the same technique as in Section 2 we obtain then in configuration space the equations

$$[\mathbf{1}_{\varrho_i} + \hat{\mathbf{K}}_{\varrho_i}]\, \varphi_{\varrho_i}(\mathfrak{K}_i) = 0\,, \quad 1 \leqq i \leqq n\,, \tag{4.4}$$

and

$$[\mathbf{1}_{\varrho} + \hat{\mathbf{K}}_{\varrho}]\, \varphi_{\varrho}{}^{(\pm)}(\mathfrak{K}_1 \ldots \mathfrak{K}_n) = 0\,. \tag{4.5}$$

By (4.4) it follows immediately that instead of (2.23) now the equation

$$[\mathbf{1}_{\varrho_1} + \hat{\mathbf{K}}_{\varrho_1}] \otimes \ldots \otimes [\mathbf{1}_{\varrho_n} + \hat{\mathbf{K}}_{\varrho_n}] \varphi_{\varrho_1} \ldots \varphi_{\varrho_n} = 0 \tag{4.6}$$

is satisfied by the product functions of the cluster states and also

$$Q_\varkappa [\mathbf{1}_{\varrho_1} + \hat{\mathbf{K}}_{\varrho_1}] \otimes \ldots \otimes [\mathbf{1}_{\varrho_n} + \hat{\mathbf{K}}_{\varrho_n}] Q_\varkappa{}^{-1} Q_\varkappa \varphi_{\varrho_1} \ldots \varphi_{\varrho_n} = 0\,, \quad 1 \leqq \varkappa \leqq L\,, \tag{4.7}$$

is valid. To derive the channel equations we observe

$$[\mathbf{1}_{\varrho_1} + \hat{\mathbf{K}}_{\varrho_1}] \otimes \ldots \otimes [\mathbf{1}_{\varrho_n} + \hat{\mathbf{K}}_{\varrho_n}] = \mathbf{1}_{\varrho} + \sum_{r=1}^{n} P' \sum_{\alpha_1 \ldots \alpha_r} \mathbf{K}_{\varrho_{\alpha_1}} \otimes \ldots \otimes \mathbf{K}_{\varrho_{\alpha_r}} =: \mathbf{1}_{\varrho} + \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n)\,. \tag{4.8}$$

Therefore (4.7) can be written generally

$$[\mathbf{1}_{\varrho} + Q_\varkappa \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n) Q_\varkappa{}^{-1}]\, Q_\varkappa \varphi_{\varrho_1} \ldots \varphi_{\varrho_n} = 0\,, \quad 1 \leqq \varkappa \leqq L\,. \tag{4.9}$$

Defining the channel interaction operator

$$\hat{\mathbf{V}}_{\varrho}(\varkappa) := \mathbf{K}_{\varrho} - Q_\varkappa \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n) Q_\varkappa{}^{-1},\ 1 \leqq \varkappa \leqq L\,, \tag{4.10}$$

Eq. (4.5) reads now especially for $\varphi_{\varrho}{}^{(\pm)}(\varkappa, \mathfrak{K}_1 \ldots \mathfrak{K}_n)$

$$[\mathbf{1}_{\varrho} + Q_\varkappa \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n) Q_\varkappa{}^{-1} + \hat{\mathbf{V}}_{\varrho}(\varkappa)]\, \varphi_{\varrho}{}^{(\pm)}(\varkappa, \mathfrak{K}_1 \ldots \mathfrak{K}_n) = 0\,,\ 1 \leqq \varkappa \leqq L\,, \tag{4.11}$$

and leads to the channel equation

$$\varphi_{\varrho}{}^{(\pm)}(\varkappa, \mathfrak{K}_1 \ldots \mathfrak{K}_n) = Q_\varkappa \varphi_{\varrho_1}(\mathfrak{K}_1) \ldots \varphi_{\varrho_n}(\mathfrak{K}_n) + \lim_{\gamma \to 0} [\mathbf{1}_{\varrho} + Q_\varkappa \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n) Q_\varkappa{}^{-1} \pm i\gamma]^{-1} \cdot \hat{\mathbf{V}}_{\varrho}(\varkappa)\, \varphi_{\varrho}{}^{(\pm)}(\varkappa, \mathfrak{K}_1 \ldots \mathfrak{K}_n)\,,\ 1 \leqq \varkappa \leqq L\,. \tag{4.12}$$

Observing (4.8) we obtain for infinitesimal small γ

$$[\mathbf{1}_{\varrho} + \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n) \pm i\gamma]^{-1} = [\mathbf{1}_{\varrho_1} + \hat{\mathbf{K}}_{\varrho_1} \pm i\gamma/n]^{-1} \otimes \ldots \otimes [\mathbf{1}_{\varrho_n} + \hat{\mathbf{K}}_{\varrho_n} \pm i\gamma/n]^{-1} \tag{4.13}$$

and

$$[\mathbf{1}_{\varrho} + Q_\varkappa \hat{\mathbf{K}}_{\varrho}(\varrho_1 \ldots \varrho_n) Q_\varkappa{}^{-1} \pm i\gamma]^{-1} = Q_\varkappa [\mathbf{1}_{\varrho_1} + \hat{\mathbf{K}}_{\varrho_1} \pm i\gamma/n]^{-1} \otimes \ldots \otimes [\mathbf{1}_{\varrho_n} + \hat{\mathbf{K}}_{\varrho_n} \pm i\gamma/n]^{-1} Q_\varkappa{}^{-1},\ 1 \leqq \varkappa \leqq L\,. \tag{4.14}$$

So finally the channel equations read

$$\varphi_{\varrho}{}^{(\pm)}(\varkappa, \mathfrak{K}_1 \ldots \mathfrak{K}_n) = Q_\varkappa \varphi_{\varrho_1}(\mathfrak{K}_1) \ldots \varphi_{\varrho_n}(\mathfrak{K}_n) + \lim_{\gamma \to 0} Q_\varkappa [\mathbf{1}_{\varrho_1} + \hat{\mathbf{K}}_{\varrho_1} \pm i\gamma/n]^{-1} \otimes \ldots \otimes [\mathbf{1}_{\varrho_n} + \hat{\mathbf{K}}_{\varrho_n} \pm i\gamma/n]^{-1} Q_\varkappa{}^{-1} \mathbf{V}_{\varrho}(\varkappa) \varphi_{\varrho}{}^{(\pm)}(\varkappa, \mathfrak{K}_1 \ldots \mathfrak{K}_n),\ 1 \leqq \varkappa \leqq L\,. \tag{4.15}$$

All other arguments run on the pattern. Whether the channel Eqs. (4.15) or (2.27) have to be prefered cannot be decided generally. It depends on a thorough analysis of the graph structure of $\mathbf{V}_{\varrho}(\varkappa)$ resp. $\hat{\mathbf{V}}_{\varrho}(\varkappa)$ and of an additional discussion of the boundary conditions. This will be done in following papers on the subject.

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[1] W. Heisenberg, An Introduction to the Unified Theory of Elementary Particles, Wiley and Sons, London 1967.
[2] H. Stumpf, Quanten und Felder, Ed. H. P. Dürr, Vieweg u. Sohn, Braunschweig 1971, S. 189.
[3] H. Stumpf, Z. Naturforsch. **27 a**, 1058 [1972].
[4] H. Stumpf, Acta Phys. Austriaca, Suppl. **IX**, 195 [1972].
[5] H. Stumpf, Z. Naturforsch. **26 a**, 623 [1971].
[6] H. P. Dürr and F. Wagner, Nuovo Cim. **46**, 223 [1966].
[7] H. Rampacher, H. Stumpf, and F. Wagner, Fortschr. Phys. **13**, 385 [1965].
[8] K. Dammeier, Z. Naturforsch. **27 a**, 1042 [1972].
[9] M. L. Goldberger and K. M. Watson, Cossision Theory, J. Wiley, New York 1965.